GENUS COMPUTATION OF GLOBAL FUNCTION FIELDS

Jens-Dietrich Bauch

Universidad Autònoma de Barcelona

bauch@mat.uab.cat

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Genus

Let F/k be a function field and denote $\mathcal{D}_F$ the set of divisors of F. The genus of F is the non-negative number

$$g := \max\{\deg A - \dim A + 1 \mid A \in \mathcal{D}_F\}.$$

Defining polynomial

Let $f(t,x) = x^n + a_1(t)x^{n-1} + \cdots + a_n(t)$ be the defining polynomial of F/k. We set:

- $C_f := \max\{\lceil \deg a_i(t)/i \rceil \mid 1 \leq i \leq n\}$
- $f_\infty := t^{-nC_f} f(t, t^{C_f} x) \in k[t^{-1}, x]$
- $A := k[t]$
- $A_\infty := k[t^{-1}]_{(t^{-1})}$

Orders

Let $\theta \in F$ with $f(t, \theta) = 0$.

- $A[\theta]$ finite equation order.
- $\mathcal{O}_F := \mathrm{Cl}(A, F)$ finite maximal order.
- $A_\infty[\theta_\infty]$ infinite equation order, where $\theta_\infty := \theta / t^{C_f}$.
- $\mathcal{O}_{F,\infty} := \mathrm{Cl}(A_\infty, F)$ infinite maximal order.

Denote by k_0 the full constant field of F/k.

Theorem

The genus may be computed as:

$$g = \frac{[k_0 : k] - n - \deg[\mathcal{O}_F : A[\theta]] + \deg[\mathcal{O}_{F,\infty} : A_\infty[\theta_\infty]] + C_f n(n-1)/2}{[k_0 : k]}$$

If $[k_0 : k] = 1$*, we obtain*

$$g = 1 - n - \deg[\mathcal{O}_F : A[\theta]] + \deg[\mathcal{O}_{F,\infty} : A_\infty[\theta_\infty]] + C_f n(n-1)/2$$

Montes Algorithm

Input:
-$f(t,x)$ defining polynomial of a global function field F/k.
-An irreducible polynomial $p(t) \in A$.
Ouput:
Non-negative integer $v_{p(t)}([\mathcal{O}_F : A[\theta]])$

Montes Algorithm

Input:
-f_∞ defining polynomial of a global function field F/k.
-The irreducible polynomial $1/t \in A_\infty$.
Ouput:
Non-negative integer $v_\infty([\mathcal{O}_{F,\infty} : A_\infty[\theta_\infty]])$

Genus Algorithm

Input: A global function field F/k with defining polynomial f of degree n.
Ouput: Genus g of F.

- $f_\infty \leftarrow t^{-C_f n} f(t, t^{C_f} x)$
- $\mathit{FiniteIndex} \leftarrow 0$
- Factorize $\mathrm{Disc}(f)$
- For all irreducible polynomials $p(t)$ with $v_{p(t)}(\mathrm{Disc}(f)) \geq 2$
  - $\mathrm{ind}_{p(t)} \leftarrow \texttt{Montes-algorithm}(f, p(t))$
  - $\mathit{FiniteIndex} \leftarrow \mathit{FiniteIndex} + \deg(p(t)) \cdot \mathrm{ind}_{p(t)}$
- $\mathit{InfiniteIndex} \leftarrow \texttt{Montes-algorithm}(f_\infty, 1/t)$
- **return** $1 - n - \mathit{FiniteIndex} - \mathit{InfiniteIndex} + C_f n(n-1)/2$

Complexity

Theorem

Let F/k be a function field over the finite field k with q elements and with defining polynomial f of degree n. Then, the algorithm needs at most

$$O(n^{5+\epsilon} C_f^{2+\epsilon} \log(q))$$

operations in k to determine the genus of F.

We consider the function field F/k of genus g, with defining polynomial $f_l(t,x) \in k[t,x]$.

$f_1(t,x) = x^2 + t$
$f_2(t,x) = f_1(x)^2 + (t-1)t^3 x$
$f_3(t,x) = f_2(x)^3 + t^{11}$
$f_4(t,x) = f_3(x)^3 + t^{29} x f_2(x)$
$f_5(t,x) = f_4(x)^2 + (t-1)t^{42} x f_1(x) f_3(x)^2$
$f_6(t,x) = f_5(x)^2 + t^{88} x f_3(x) f_4(x)$

l	g	deg f_l	I.C.	Genus Algo	Magma
1	0	2	0.0	0.0	0.0
2	3	4	0.0	0.01	0.01
3	9	12	0.01	0.02	1.66
4	40	36	0.09	0.12	707.06
5	133	72	1.37	1.61	60125.24
6	329	144	22.06	24.67	–

The End