# A recipe for writing down generators for modular units.

**Maarten Derickx**[1] Mark van Hoeij[2]

[1]Universiteit Leiden

[2]Florida State University

ANTS X - grump session

# Equations for $Y_1(N)$

- $R := \mathbb{Z}[b, c, \frac{1}{\Delta}]$ with $\Delta := -b^3(16b^2 + (8c^2 - 36c + 27)b + (c-1)c^3)$
- $E/R$ ell. crv given by $Y^2 + cXY + bY = X^3 + bX^2$
- $P := (0:0:1)$
- Let $\Phi_N, \Psi_N, \Omega_N \in R$ be s.t. $(\Phi_N \Psi_N : \Omega_N : \Psi_N^3) = NP$

The equation $\Psi_N = 0$ sais $P$ has order dividing $N$. Define $F_N$ by removing form $\Psi_N$ all factors coming from some $\Psi_d$ with $d|N$.

- $Y_1(N)_{\mathbb{Z}[1/N]} = \operatorname{Spec} R[1/N]/F_N$
- hence also $\mathbb{Q}(X_1(N)) = \mathbb{Q}(b)[c]/F_N$.
- if $N \neq M$ consider $F_M \in \mathbb{Q}(X_1(N))^*$ then $\operatorname{supp}\operatorname{div} F_M \subset Cusps$.
- if $N \nmid M$ consider $\Psi_M \in \mathbb{Q}(X_1(N))^*$ then $\operatorname{supp}\operatorname{div} \Psi_M \subset Cusps$.
- the above means that $F_M$ and $\Psi_M$ are modular unit.

# Modular units over $\mathbb{Q}$

Define the modular units over $\mathbb{Q}$ as follows.

$$\mathcal{F}_1(N) := \{f \in \mathbb{Q}(X_1(N))^* \,|\, \text{supp div } f \subset \mathit{Cusps}\}/\mathbb{Q}^*$$
$$\mathcal{F}_1'(N) := \langle \Delta, b, F_4, \ldots, F_{\lfloor N/2 \rfloor + 1} \rangle = \langle \Delta, b, \Psi_4, \ldots, \Psi_{\lfloor N/2 \rfloor + 1} \rangle \subset \mathcal{F}_1(N)$$

- $X_1(N)$ has $\lfloor N/2 \rfloor + 1$ $\mathrm{Gal}(Q(\zeta_N)/Q)$ orbits of cusps
- The subgroup $C_1(N) \subset \mathrm{Pic}^0(X_1(N))(Q(\zeta_N))$ generated by cuspidal divisors is torsion.
- So $\mathcal{F}_1(N)$ is a lattice of rank $\lfloor N/2 \rfloor$

We verified using computer computations that for all $4 \leq N \leq 70$.

$$\mathcal{F}_1'(N) = \mathcal{F}_1(N)$$

Is there a theoretical explanation? What are the relation between the $F_N$ and the siegel functions?